AN ESTIMATE OF DIFFERENCE BETWEEN THE SZASZ - INVERSE BETA OPERATORS AND THE SZASZ - MIRAKJAN OPERATORS

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***Abstract**: In this paper we give an estimate of difference between the Szasz-Inverse Beta operators and the Szasz-Mirakjan operators.*

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1. INTRODUCTION

We deal in this paper, with an approximation operator linear positive, namely Szasz-Inverse Beta operator, which is a mixed summation-integral type operator and we give an estimate, in the terms of modulus of continuity, of the difference between this operator and the Szasz-Mirakjan operator.

2. PROBABILISTIC REPRESENTATION OF SOME OPERATORS

In our paper [1] we consider a probabilistic representation of the Szasz - Inverse Beta operators, which were defined and investigated by V. Gupta, M.A. Noor, [4] and iterative constructions of these operators were studied recently by Z. Finta, N.K. Govil, V. Gupta [3] :

$$L_t(f;x) =$$
$$= e^{-tx}f(0) + \sum_{k=1}^{\infty} s_{t,k}(x)\int_0^{\infty} b_{t,k}(u)f(u)du =$$
$$= \int_0^{\infty} J_t(u;x)f(u)du, \quad x \geq 0 \tag{1}$$

with

$$s_{t,k}(x) = e^{-tx}\frac{(tk)^k}{k!} \tag{2}$$
$$t > 0,\ x \geq 0, k \in N \cup \{0\}$$

$$b_{t,k}(u) = \frac{1}{B(k,t+1)} \cdot \frac{u^{k-1}}{(1+u)^{t+k+1}} \tag{3}$$
$$u > 0,\ t > 0$$

$$B(k,t+1) = \int_0^{\infty} \frac{u^{k-1}}{(1+u)^{t+k+1}} du \tag{4}$$

being Inverse- Beta function,

$$J_t(u;x) = e^{-tx}\delta(u) + \sum_{k=1}^{\infty} s_{t,k}(s)b_{t,k}(u) \tag{5}$$

$\delta(u)$ being the Dirac's delta function, for which $\int_0^{\infty}\delta(u)f(u)du = f(0)$.

So, these operators are represented as, the mean value of the random variable $\frac{U_{N(t\ x)}}{V_{t+1}}$ which has the probability density function $J_t(u;x)$ defined as (5) :

$$L_t(f;x) = E[f(Z_{t\ x})] = E\left[f\left(\frac{U_{N(t\ x)}}{V_{t+1}}\right)\right] \tag{6}$$
$$t > 0, \quad x \geq 0$$

with $\{N(t): t \geq 0\}$ a standard Poisson process and $\{U_t : t \geq 0\}$, $\{V_t : t \geq 0\}$ two mutually independent Gamma processes defined all on the same probability space.

Note that, the Poisson process is a stochastic process starting at the origin, having stationary independent increments with probability:

$$P(N(t)=k) = \frac{e^{-t}t^k}{k!}, \; t \geq 0, \; k \in N \cup \{0\} \quad (7)$$

and the Gamma process is a stochastic process starting at the origin $(U_0 = 0)$, having stationary independent increments and such that, for t > 0, U_t has the Gamma probability density function:

$$d_t(u) = \begin{cases} \frac{u^{t-1}e^{-u}}{\Gamma(t)}, & t > 0, \, u > 0 \\ 0, & u \leq 0 \end{cases} \quad (8)$$

and without loss of generality [5] it can assumed that $\{U_t : t \geq 0\}$ and $\{V_t : t \geq 0\}$ for each t > 0 have a.s. no decreasing right-continuous paths.

On these operators it is know, that:

$$L_t(e_i; x) = e_i(x), \; i = \overline{0,1}, \; x \geq 0$$

$$L_t(e_2; x) = \frac{t}{t-1}x^2 + \frac{2}{t-1}x, \; t > 1, \; x \geq 0$$

$$L_t(e_2 - x^2; x) = L_t((e_1 - x)^2; x) =$$

$$= D^2\left[\frac{U_{N(t\,x)}}{V_{t+1}}\right] = E\left[\left(\frac{U_{N(t\,x)}}{V_{t+1}} - x\right)^2\right] =$$

$$= \frac{x(2+x)}{t-1}, \; t > 1, \; x \geq 0$$

and

$$L_t(f; x) = (S_t \circ T_t)(f; x) = S_t(T_t)(f; x),$$

$$t > 0, \; x \geq 0 \quad (9)$$

with

$$\begin{cases} T_t(f; x) = \frac{1}{B(tx, t+1)} \int_0^\infty \frac{u^{t\,x-1}}{(1+u)^{t\,x+t+1}} f(u)du \\ T_t(f;0) = f(0) \end{cases}$$

$\Rightarrow$

$$\begin{cases} T_t(f; x) = \int_0^\infty f(u) b_{t\,x,t+1}(u)du, \; t > 0, \; x > 0 \\ T_t(f;0) = f(0) \end{cases} \quad (10)$$

the Inverse-Beta operators or the Stancu's operators of second kind [6] having $b_{t\,x,t+1}(u)$ as (3).

The Inverse-Beta operators $T_t(f; x)$ preserve the affine functions on $[0,\infty)$:

$$T_t(e_0; x) = e_0(x) = 1$$

$$T_t(e_1; x) = e_1(x) = x$$

$$T_t(e_2; x) = x^2 + \frac{x(x+1)}{t-1}, \; t > 1$$

Using the classical estimate for the linear positive operators:

$$|(Lf)(x) - f(x)| \leq$$

$$\leq \left(1 + \delta^{-2} L(e_1 - xe_0)^2(x)\right)\omega(f; \delta),$$

$$f \in C_B(I), \; I \subset R, \; \delta > 0$$

we have for these operators:

$$|T_t(f; x) - f(x)| \leq \left(1 + \delta^{-2}\frac{x(x+1)}{t-1}\right)\omega(f; \delta),$$

$$(\forall) \; f \in C_B[0, +\infty), \; t > 1 \quad (11)$$

These operators can be probabilistic represented as the mean value of the random variable $f(W_{t\,x,\,t+1}) = f\left(\frac{U_{t\,x}}{V_{t+1}}\right)$, where $U_{t\,x}, V_{t+1}$ are two independent random variables, having Gamma distribution with density $d_{t\,x}(u)$ respectively $d_{t+1}(u)$ defined as (8):

$$\begin{cases} T_t(f; x) = E[f(W_{t\,x,t+1})] = E\left[f\left(\frac{U_{t\,x}}{V_{t+1}}\right)\right] \\ \qquad t > 0, x > 0 \\ T_t(f;0) = f(0) \end{cases}$$

The well known Szasz-Mirakjan's operators:

$$S_t(f;x)=\sum_{k=1}^{\infty}s_{t,k}(k)f\left(\frac{k}{t}\right) \tag{12}$$

with $s_{t,k}(x)$ defined as (2) can be represented as the mean value of the random variable $f\left(\frac{N(t\ x)}{t}\right)$, $t>0,\ x\geq 0$, where the random variable $N(t\ x)$ has the Poisson distribution and take the value k with probability $s_{t,k}(x)$ as (2).

So, these operators:

$$S_t(f;x)=E\left[f\left(\frac{N(t\ x)}{t}\right)\right],$$

$$t>0,\ x\geq 0 \tag{13}$$

are well defined, if f is a real measurable function on $[0,\infty)$ such that:

$$E\left[\left|f\left(\frac{N(t\ x)}{t}\right)\right|\right]<\infty \text{ for each } t>0$$

and

$$S_t(e_0;x)=e_0(x)=1$$

$$S_t(e_1;x)=e_1(x)=E\left[\frac{N(t\ x)}{t}\right]=x$$

$$S_t(e_2;x)=E\left[\left(\frac{N(t\ x)}{t}\right)^2\right]=x^2+\frac{x}{t}$$

$$t>0,\ x\geq 0 \tag{14}$$

Using our paper [1, Th.3.2] and a result of De la Cal, J., Carcamo J., [2] we have for all convex functions in the domain of these operators $\mathcal{L}_{cx}[0,\infty)$ that,

$$L_t f\geq S_t f,\ f\in\mathcal{L}_{cx}[0,\infty) \tag{15}$$

For this, in the next section we give an estimate for the difference $\left|L_t(f;x)-S_t(f;x)\right|$.

3. AN ESTIMATE OF DIFFERENCE $L_t(f;x)-S_t(f;x)$

In view of (15) using the representation (9) for the Szasz-Inverse Beta operators, the estimate (11) for the Inverse-Beta operators and the Szasz-Mirakjan`s properties (14) we have:

Theorem 3.1 If $f\in C_B[0,\infty)\cap\mathcal{L}_{cx}[0,+\infty)$ then for every $x\in[0,+\infty)$ and $t>1$

$$\left|L_t(f;x)-S_t(f;x)\right|\leq$$

$$\leq\left(1+\delta^{-2}\left(\frac{x(x+1)}{t-1}+\frac{x}{t(t-1)}\right)\right)\omega(f;\delta)$$

Proof.

$$\left|L_t(f;x)-S_t(f;x)\right|=\left|S_t(T_t(f;x))-S_t(f;x)\right|\leq$$

$$\leq\sum_{k=1}^{\infty}s_{t,k}(x)\left|T_t f\left(\frac{k}{t}\right)-f\left(\frac{k}{t}\right)\right|\leq$$

$$\leq\sum_{k=1}^{\infty}s_{t,k}(x)\left(1+\delta^{-2}\frac{\frac{k}{t}\left(\frac{k}{t}+1\right)}{t-1}\right)\omega(f;\delta)\leq$$

$$\leq\sum_{k=1}^{\infty}e^{-tx}\frac{(tx)^k}{k!}\left(1+\delta^{-2}\frac{k(k+1)}{t^2(t-1)}\right)\omega(f;\delta)\leq$$

$$\leq\left(1+\frac{\delta^{-2}}{t^2(t-1)}\left((tx)^2+tx+t^2x\right)\right)\omega(f;\delta)$$

$$\left|L_t(f;x)-S_t(f;x)\right|\leq$$

$$\leq\left(1+\delta^{-2}\left(\frac{x(x+1)}{t-1}+\frac{x}{t(t-1)}\right)\right)\omega(f;\delta)$$

For

$$\delta=\frac{1}{\sqrt{t-1}},\ t>1,\ x\geq 0,\ f\in C_B[0,+\infty)$$

we obtain

$$\left|L_t(f;x)-S_t(f;x)\right|\leq$$

$$\leq\left(1+\left(x(x+1)+\frac{x}{t}\right)\right)\omega\left(f;\frac{1}{\sqrt{t-1}}\right)$$

Using the probabilistic representation (6), (13) of these operators, we can to give an estimate with the aid of the variances of the random variables $\frac{U_{tx}}{V_{t+1}}$ and $\frac{N(tx)}{t}$.

So that, result for $t > 1,\ x \ge 0,\ \delta > 0$

$$\left| E\left[f\left(\frac{U_{N(tx)}}{V_{t+1}} \right) \right] - E\left[f\left(\frac{N(tx)}{t} \right) \right] \right| \le$$

$$\le \left(1 + \delta^{-2} \left(D^2\left(\frac{U_{tx}}{V_{t+1}} \right) + \frac{1}{t-1} D^2\left(\frac{N(tx)}{t} \right) \right) \right) \cdot$$

$$\cdot\, \omega(f;\delta)$$

4. CONCLUSIONS

This study is interesting because presents an estimate of a difference between the images of the same function with the aid of two different operators but one of them is a mixture of the other operator and another one.

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